Yau Mathematical Competition 2018
Probability and Statistics Individual

Problem 1 (Probabability) Suppose that for each n, the $(n \times n)$ random matrix X^n has the uniform distribution on the orthogonal group $O(n)$.

(1) What is the distribution of the first row vector

$$X_1^n = (X_{11}^n, X_{12}^n, \dots X_{1n}^n)?$$

(2) Show that in distribution

$$X_{11}^n \sim \frac{Z_1}{\sqrt{\sum_{i=1}^n Z_i^2}},$$

where Z_i are independent, identically distributed random variables with the standard normal distribution.

(3) Find the limit in distribution of the random variables $\sqrt{n}X_{11}^n$ as $n \to \infty$.

Problem 2 (Statistics) Suppose we toss an unbiased coin and record K_1, the number of tosses needed to obtain the first head. Then, we draw X_1 from a normal distribution with mean $K_1\mu$ and variance $K_1\sigma^2$, and record the pair (K_1, X_1). By repeating the experiment $n-1$ times, we obtain the pairs $(K_2, X_2), \dots, (K_n, X_n)$. Using all the n data pairs $(K_1, X_1), \dots, (K_n, X_n)$:

(1) How would you best estimate μ and σ^2?

(2) Can you give a 95% confidence interval for μ?